

The Average Age of a Stationary Population and the Age x at which $x = e(x)$

1. Definitions and Properties of Life Table Functions

THE principal life table functions are conveniently defined as :

- (a) $l(x)$, the proportions of survivors from birth to age x , such that $l(0) = 1$;
- (b) $\mu(x) = -dl(x)/dx/l(x)$, the force of mortality at age x ;
- (c) $T(x) = \int_x^a l(a) da$ where a is the upper age limit at which $l(a) = 0$ and
- (d) $e(x) = T(x)/l(x)$, the expectation of life at age x .

The following properties of the functions may be noted.

- (a') $dl(x)/dx$ is uniformly negative, that is to say, $l(x)$ is a **monotonically** declining function. Further, the curve of $l(x)$ has a point of inflection, i.e., $d^2l(x)/dx^2$ equals zero for some $x = x'$. This results from the lower risk of death usually in the age interval 10-15, compared to other ages (Mitra, 1977).
- (b') The force of mortality assumes its lowest value, that is $d\mu(x)/dx = 0$ for some $x = x$, also in the age interval 10-15 (Mitra, *ibid.*).
- (c) $dT(x)/dx = -l(x)$.
- (d') $e(x)$ assumes its maximum value, that is $de(x)/dx = 0$ for some $x = \hat{x}$, usually in the age interval 0-5.

From (d), $T(x) = l(x)e(x)$, so that

$$\begin{aligned} dT(x)/dx &= e(x)dl(x)/dx + l(x)de(x)/dx \\ &= -l(x)[e(x)\mu(x) - de(x)/dx] \end{aligned} \quad (1)$$

because of (b). Equating (1) with (c) and solving

$$de(x)/dx = e(x)\mu(x) - 1. \quad (2)$$

The **maximum** life expectancy corresponds to age \hat{x} , where

$$e(x) = 1/\mu(\hat{x}); \quad (3)$$

It was shown earlier (Mitra, *ibid.*) that

$$\hat{x} < \tilde{x} < x'$$

2. Average Age of a Stationary Population

Interpreting the life table functions $l(x)dx$ and $T(x)$ as the population sizes in the age interval $(x, x + dx)$ and at ages x and above respectively in the corresponding stationary population, the average age can be defined as (Mitra, 1965)

$$m = \frac{\int_0^{\infty} xl(x)dx}{\int_0^{\infty} l(x)dx} = \frac{\int_0^{\infty} xl(x)dx}{T(0)}. \quad (4)$$

Integrating by parts, the integral

$$\int_0^{\infty} xl(x)dx = - \int_0^{\infty} xdT(x) = xT(x) \Big|_0^{\infty} + \int_0^{\infty} T(x)dx = \int_0^{\infty} T(x)dx \quad (5)$$

as $T(\infty) = 0$. Since $T(x) = l(x)e(x)$ due to (d), (5) can be rewritten as

$$\int_0^{\infty} xl(x)dx = \int_0^{\infty} T(x)dx = \int_0^{\infty} l(x)e(x)dx \quad (6)$$

Accordingly, (4) can be expressed in two different ways, **namely**,

$$m = \frac{\int_0^{\infty} xl(x)dx}{T(0)} = \frac{\int_0^{\infty} l(x)e(x)dx}{T(0)}. \quad (7)$$

In **last** expression in (7) can be interpreted as the average number of years that remains to be lived in the stationary population. Thus, the total population $T(0)$ that, on an average, is m years old can expect to live for an additional m number of years, again from the perspective of an arithmetic average.

3. Expectation of Life at Age m

In terms of the statistical concepts of the expectation of a function $f(x)$ which is traditionally expressed as $E[f(x)]$, we may write (7) as

$$m = E[e(x)]. \quad (8)$$

It is well known that under certain simplifying assumptions of continuity and differentiability which holds for the life expectancy function $e(x)$, one can write (8) as

$$m = E[e(x)] = e(m) + \int_{r=2}^{\infty} (x - m)^r e^{(r)}(m) / r! \quad (9)$$

where $e^{(r)}(m)$ is the r th derivative of $e(x)$ at $x = m$.

Thus as a first approximation, the life expectancy at the average age m is also m , a result which, intuitively at least, seems to suggest itself. The average age can therefore be approximately determined as the point of intersection \bar{x} of the line $y = x$ and the curve $y = e(x)$ (at which $\bar{x} = e(\bar{x})$). The parameters m , $e(m)$, \bar{x} together with $e(0)$, $l(x)$ and $T(x)$, obtained from the North region model life tables (Coale and Demeny, 1966) covering a wide range of variation of patterns of mortality are presented in Table 1.

It may be noted that the functions m , $e(m)$ and \bar{x} are quite comparable with one another and that \bar{x} lies between the other two. As one would expect, all of these parameters increase with the expectation of life at birth.

4. The Trends of $e(0)$ and m

It may be seen from Table 1 that the life expectancy is smaller than the average age under high mortality conditions. With the improvement in mortality, the difference between the two reverses its sign and the life expectancy is almost twice as large as the average age under the lowest mortality condition. (See Figure 1).

In fact, it can be shown that the ratio of $e(0)/m$ has an upper limit of 2. To show that we begin by expressing the frequency function of deaths as $l(x)\mu(x)$ and confirm that

$$\int_0^a l(x)\mu(x)dx = - \int_0^a dl(x) = l(0) - l(a) = l(0) \quad (10)$$

Next we evaluate the mean \bar{d} and the variance σ_d^2 of the distribution of death as

$$\bar{d} = \frac{\int_0^a x l(x)\mu(x)dx}{\int_0^a l(x)\mu(x)dx} = - \frac{\int_0^a x dl(x)}{\int_0^a dl(x)} = - \frac{x l(x)}{l(0)} + \int_0^a l(x)dx = e(0) \quad (11)$$

TABLE 1—TRENDS OF A FEW SELECTED LIFE TABLE FUNCTIONS BY LEVELS OF MORTALITY (North Region Model Life Tables for Females; Coale and Demeny, *ibid.*)

Model Number	$e(0)$	m	$e(m)$	$\bar{x} = e(\bar{x})$	$l(\bar{x})$	$T(\bar{x})$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
2	22.5	26.7	27.6	27.3	.3600	9.835
4	27.5	28.4	29.0	28.8	.4256	12.257
6	32.5	30.0	30.3	30.2	.4880	14.710
8	37.5	31.4	31.5	31.5	.5475	17.212
10	42.5	32.6	32.6	32.6	.6054	19.743
12	47.5	33.8	33.7	33.8	.6620	22.317
14	52.5	34.9	34.5	34.7	.7204	24.933
16	57.5	35.9	35.3	35.6	.7768	27.602
18	62.5	36.9	36.1	36.5	.8304	30.325
20	67.5	38.0	37.0	37.4	.8815	33.091
22	72.5	39.0	37.9	38.4	.9308	35.785
24	77.5	40.4	39.2	39.7	.9681	38.578

as is quite well known and then

$$\sigma_d^2 = \int_0^{\alpha} x^2 l(x) \mu(x) dx - e^*(0)$$

which simplifies as before to

$$\sigma_d^2 = e(0) [2m - e(0)]. \quad (12)$$

Since the variance cannot be negative it follows that

$$e(0) < 2m. \quad (13)$$

It may be noted that the upper limit is reached when everyone dies at age α , the

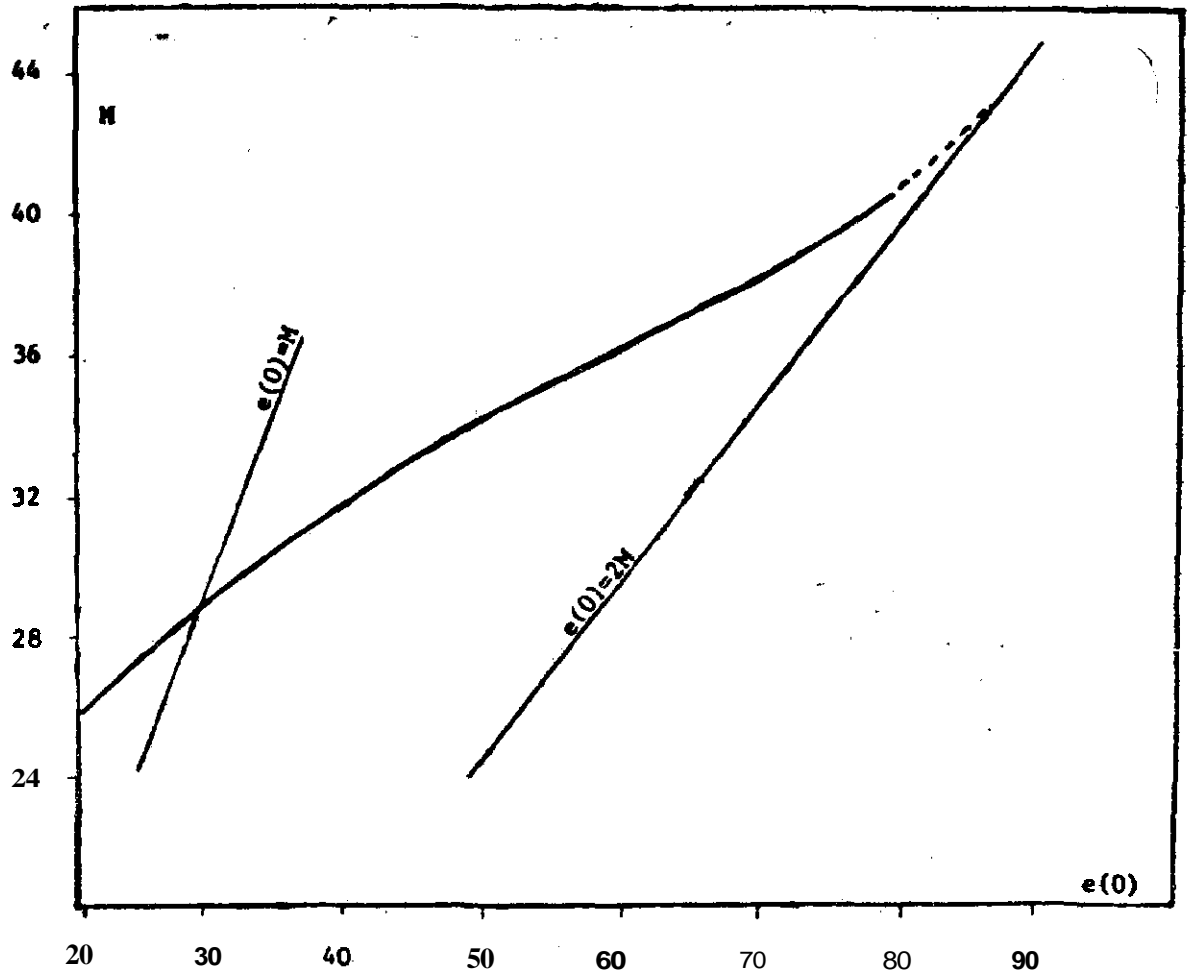


Fig. 1. The variation of the average age of the stationary population M with the expectation of life at birth $e(0)$. (North female life tables).

end of the **life span**, resulting in a rectangular distribution of $l(x)$. In the **following** we shall examine a few more interesting properties of the life table functions and in the **process**, obtain a boundary condition for $e(0)$ which, at the upper end, is no wider than (13).

5. The Curves of $T(x)$ and $xl(x)$

The definite integrals of the functions $T(x)$ and $xl(x)$ over the entire age range were earlier shown to be equal (see equation 6). However, the curves that they generate are quite different and have some interesting features.

The derivative of $T(x)$ is $-l(x)$ which is uniformly **negative**, assumes the value of -1 at age 0, slowly decreases **thereafter** and becomes equal to 0 at the highest age α . Thus the curve of $T(x)$ is concave upwards and declines from a maximum value of $T(0)$ at $x = 0$ to its lowest value of 0 at $x = \alpha$.

The function $xl(x)$ on the other hand, assumes its lowest value of 0 at both extremes of the age interval. Its derivative

$$d[xl(x)]/dx = l(x) [1 - x\mu(x)] \tag{14}$$

indicates a maximum value of the function at some **age**, say x'' where

$$x'' = 1/\mu(x''). \tag{05}$$

Because of the nature of the functions $l(x)$ and $\mu(x)$, the function of $xl(x)$ has **only** one maximum and the curve of the function is bell shaped.

The general nature of the curves of $T(x)$ and $xl(x)$ together with those of $y = x$ and $y = e(x)$ may be studied from Figure 2.

A. The Point of Intersection

Interestingly enough, the x coordinate of the point of intersection between $xl(x)$ and $T(x)$ is the **same** as that between (see **section 3**) the curve of $e(x)$ and $y = x$, namely, \bar{x} , since, $\bar{x} = e(x)$ implies

$$T(\bar{x}) = \bar{x}l(\bar{x}). \tag{16}$$

It can be shown that $\bar{x} < x''$ where **the** latter age corresponds to the maximum value of $xl(x)$. This is so **because**, in the neighborhood of age x , the life expectancy is usually a declining function of age so **that**, its derivative (see equation 2) $e(x)\mu(x) - 1$ is negative at $x = \bar{x}$. Since $\bar{x} = e(x)$, we can write this derivative as

$$e(\bar{x})\mu(\bar{x}) - 1 = \bar{x}\mu(\bar{x}) - 1 < 0 \tag{17}$$

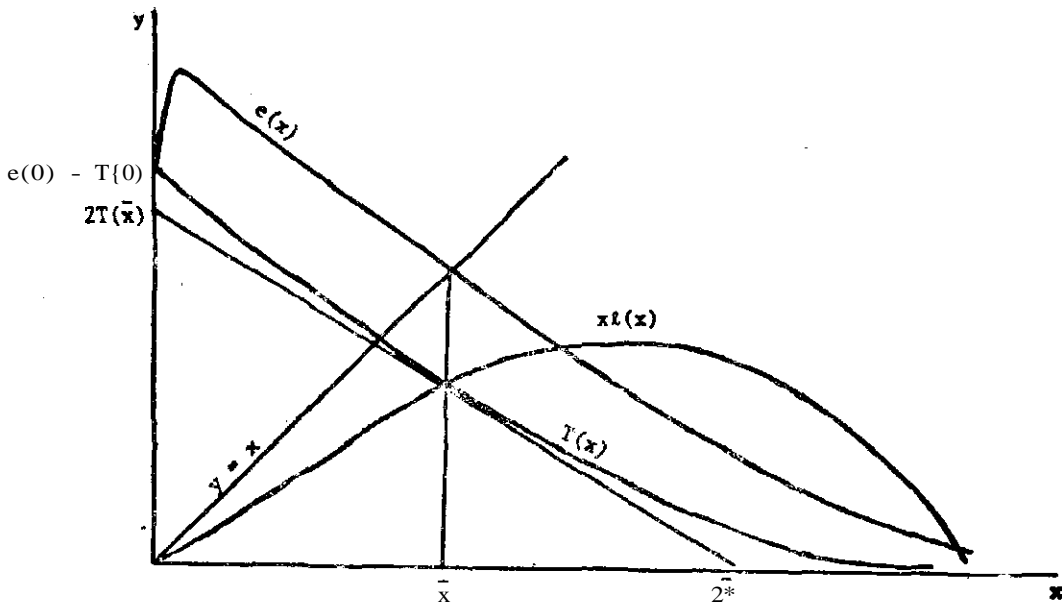


Fig. 2. The Curves of $xl(x)$, $T(x)$ and $e(x)$.

so that (14) is positive at $x = \bar{x}$ suggesting that the function $xl(x)$ is still increasing at \bar{x} . Consequently, $x < x^*$.

B. The Tangent to $T(x)$

The tangent to $T(x)$ meets the x axis at a point, the x coordinate of which (since the derivative of $T(x)$ is $-l(x)$), is

$$x + T(x)/l(x) = x + e(x). \quad (18)$$

Therefore, the tangent at $T(\bar{x})$ is $\bar{x} + e(\bar{x}) = 2\bar{x}$ which must intersect the y axis at $2T(\bar{x})$. Since the curve of $T(x)$ is concave upwards

$$2T(\bar{x}) < T(0) = e(0) \quad (19)$$

(see Col. 7 of Table 1) which means that more than half of the population is less than x years old. This is not unexpected in view of the monotonic declining nature of the $l(x)$ function.

The right angled triangle formed by the tangent at $(x, T(x))$ as the hypotenuse and the sides determined by its intersection with the x and the y axes having

lengths $x + e(x)$ and $l(x)(x + e(x))$ respectively, is **studied next**. Its area

$$A(x) = \frac{(xl(x) + T(x))(x + e(x))}{2} \quad (20)$$

is a **function** of age x and its derivative

$$\begin{aligned} dA(x)/dx &= \frac{1}{2} [-xl(x)\mu(x)(x + e(x)) + (xl(x) + T(x))e(x)\mu(x)] \\ &= \frac{1}{2} l(x)\mu(x) [e^2(x) - x^2] \end{aligned} \quad (21)$$

is positive as long as $e(x) > x$ which happens as long as $x < \bar{x}$. Since $x = e(\bar{x})$, the derivative of $A(x)$ at \bar{x} is equal to zero. It can be shown that the second derivative of $A(x)$ at this point is negative and therefore, substituting $x = \bar{x}$ in (20), we get

$$A(x) = (xl(x) + T(x))(x + e(x))/2 = 2(xT(x))/2 = 2\bar{x}^2 l(\bar{x}) \quad (22)$$

as the **maximum** value of $A(x)$.

C. The Boundaries of $e(0)/\bar{x}$

It may be seen from (20) and (22) that

$$A(0) = T^2(0)/2 < 2\bar{x}^2 l(\bar{x}). \quad (23)$$

Alternatively, we can write (23) as

$$T(0)/x < 2\sqrt{l(x)}. \quad (24)$$

A similar alternative form of (19) is

$$2l(\bar{x}) < T(0)/\bar{x}. \quad (25)$$

A combination of (24) and (25) produces the boundaries of $e(0)/\bar{x}$ (since $T(0) = e(0)$) as

$$2l(\bar{x}) < e(0)/\bar{x} < 2\sqrt{l(\bar{x})}. \quad (26)$$

It is interesting to note that the relationship between $e(0)$ and \bar{x} can at present be expressed in the form of a boundary condition. Further investigation is needed to find out the possibility or otherwise of improving this inequality relationship.

Summary

It has been observed that the average age m of a stationary population is equal to the average number of years that remains to be lived by that population. This age or this number of years has been shown as approximately equal to the age x at which the life expectancy is also equal to that age in number of years. Interesting enough, this age x can also be obtained as the x coordinate of the point of intersection of the curves of the functions $xl(x)$ and $T(x)$. Also, the tangent to the curve of $T(x)$ at this intersection, crosses the x axis at a point which corresponds to twice the age \bar{x} . A byproduct of this investigation is an interesting boundary condition for the ratio of the expectation of life at birth $c(0)$ to X . This ratio has been found to lie between $2l(x)$ and $2\sqrt{l(\bar{x})}$ where $l(\bar{x})$ is the proportion surviving from birth to the age \bar{x} .

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References

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